UNKNOTTED HOMOLOGY CLASSES
ON UNKNOTTED SURFACES IN S^3

BRUCE TRACE

ABSTRACT. Suppose F is a closed, genus g surface which is standardly embedded in S^3. Let γ denote a primitive element in $H_1(F)$ which satisfies $\theta_F(\gamma, \gamma) = 0$ where θ_F is the Seifert pairing on F. We obtain a number theoretic condition which is equivalent to γ being realizable by a curve (in F) which is unknotted in S^3. Various related observations are included.

Let F denote the closed, orientable, connected surface of genus g and view F as being standardly embedded in S^3. Throughout this paper, γ is a primitive class in $H_1(F)$, $\mathbf{Z}$ coefficients understood, which satisfies $\theta_F(\gamma, \gamma) = 0$ where θ_F denotes the Seifert pairing on F. Provided here is an additional number theoretic criterion on γ which is equivalent to γ being realizable by an unknotted circle in S^3.

More precisely, let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ constitute a symplectic basis for $H_1(F)$ such that the α_i's are all null-homologous in the closure of one component of $S^3 - F$ and the β_i's are all null-homologous in the closure of the remaining component of $S^3 - F$. Writing γ as $\gamma = \sum_{i=1}^{g}(a_i \alpha_i + b_i \beta_i)$ we establish.

Theorem 2. *There exists an embedded 2-disk D in S^3 such that $\partial D \subset F$ and $\gamma = [\partial D]$ if and only if either $\gcd(a_1, \ldots, a_g) = 1$ or $\gcd(b_1, \ldots, b_g) = 1$.*

Theorem 2 has as a consequence the following generalization of well-known results in the case $g = 1$.

Corollary 1. *Suppose $\delta = \sum_{i=1}^{g}(a_i \alpha_i + b_i \beta_i)$ is nontrivial and $\theta_F(\delta, \delta)$ is arbitrary. Then $\delta = [\partial D]$, for some embedded disk D, implies that either $\gcd(a_1, \ldots, a_g) = 1$ or $\gcd(b_1, \ldots, b_g) = 1$.*

Thus δ is realized by an unknotted circle in S^3 only if it possesses what might be termed as a *strongly primitive property*.

Actually, Theorem 2 is obtained as a corollary of

Theorem 1. *Set* $\gamma = \gamma_0$. *Then* $\gamma = [\partial D]$ *if and only if there exist classes* $\gamma_1, \ldots, \gamma_k \in H_1(F)$ *which satisfy*

(a) $\gamma_i \cdot \gamma_j = 0$ *whenever* $i, j \in \{0, 1, \ldots, k\}$,

(b) $\gamma_l + n_{l+1}\gamma_{l+1} + \cdots + n_k\gamma_k$ *is primitive in* $H_1(F)$ *for each* l *whenever* $n_{l+1}, \ldots, n_k \in \mathbf{Z}$, *and*

(c) $\gamma_l \in \mathrm{span}\{\gamma_{l+1}, \ldots, \gamma_k\}$ *in the closure of one component of* $S^3 - F$, $l \in \{0, 1, \ldots, k\}$.

In fact, if $\gamma_1, \ldots, \gamma_k \in H_1(F)$ *satisfy conditions* (a)–(c), *there exist disjointly embedded circles* $C_1, \ldots, C_k \subset F$ *such that* $\gamma_l = [C_l]$ *for each* l *and* F *can be sequentially compressed along the* C_l*'s, i.e. there exist surfaces* $F = F_k \rightsquigarrow F_{k-1} \rightsquigarrow \cdots \rightsquigarrow F_0$ *is* S^3 *where* F_l *is obtained from* F_{l+1} *by compressing* F_{l+1} *along a disk whose boundary is* C_{l+1} *for* $l = 0, 1, \ldots, k-1$.[1]

The proof of Theorem 1 occupies §§1 and 2.

In §3 we prove Theorem 2 and include some immediate applications of Theorems 1 and 2. Of these applications the most intriguing appears to be the following.

Theorem 3. *Suppose* F *is standardly embedded in* S^3 *and* genus$(F) = g$. *Suppose further that* $\{C_1', \ldots, C_g'\}$ *is a collection of disjointly embedded, setwise nonseparating curves in* F *which satisfy* $\theta_F([C_i'], [C_j']) = 0$ *for all* i *and* j. *Then there exists a second collection of disjointly embedded, setwise nonseparating, curves in* F, $\{C_1, \ldots, C_g\}$, *such that* $[C_i] \in \mathrm{span}\{[C_1'], \ldots, [C_g']\}$ *for all* i, *together with a sequence of compressions*

$$F = F_g \rightsquigarrow F_{g-1} \rightsquigarrow \cdots \rightsquigarrow F_1 \rightsquigarrow F_0$$

where F_i *is a standardly embedded surface of genus* i *and* F_i *is obtained from* F_{i+1} *by compressing* F_{i+1} *along a disk whose boundary is* C_{i+1} *for each* i. *Moreover, upon performing band connect sum operations on* C_i' *we can obtain* $\{C_1'', \ldots, C_g''\}$ *where* $[C_i''] = [C_i]$ *for each* i.

Theorem 3 has the following geometric interpretation. Let X denote the closure of one component of $S^3 - F$ and (abstractly) attach g 2-handles to X along the curves C_i'. We then obtain a closed, orientable 3-manifold M' by adding a 3-handle to $X \cup$ 2-handles. Theorem 3 tells us that a 3-manifold, M say, which looks very much like M' can be smoothly embedded in S^4. To be more precise, by performing 2-handle slides we have that

$$M' = X \cup_{\{C_i''\}} (\text{2-handles}) \cup (\text{3-handle}).$$

If we set

$$M = X \cup_{\{C_i\}} (\text{2-handles}) \cup (\text{3-handle}),$$

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[1] Condition (b) of Theorem 1 may be restated as $\{\gamma_0, \ldots, \gamma_k\}$ can be extended to a basis for $H_1(F)$. (b) as stated, however, is more natural for our purposes.

then M smoothly embeds in S^4 (consider the sequence of compressions) and M and M' differ "only" in how their 2-handles are attached. The attaching curve for the ith 2-handle attached to M and the attaching curve for the ith 2-handle attached to M' both represent the same homology class in ∂X (not just $H_1(X)$).

Theorem 3 also offers a curious insight into the Levine approach for slicing knots [L]. Indeed, if Theorem 3 remained true for Seifert surfaces (whose complements were handlebodies) then algebraically slice would imply ribbon. Of course, this is well known not to be the case [C-G]. Alternatively, slicing invariants which are finer than algebraically slice may be employed to restrict the homotopy classes within a homology class in F which can be realized as the boundary of an embedded disk. Theorem 2 can also be used for this purpose occasionally; see §4.

Prior to entering into §1, I would like to thank Martin Evans for a brief but informative conversation concerning combinatorial group theory. This conversation led to the conceptualization of Lemma 3. The paper's title is due, with thanks, to Martin Scharlemann.

1. THE NECESSARY CONDITION

In this section we assume that $\gamma = [\partial D]$. We then locate homology classes $\gamma_1, \ldots, \gamma_k \in H_1(F)$ such that $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k$ satisfy conditions (a)–(c) of Theorem 1. It is worth noting beforehand that at no point in this section do we invoke the hypothesis that F is standardly embedded in S^3. Therefore, the necessary condition remains valid on any (smoothly) embedded, closed surface in S^3.

We begin by observing the following geometric result.

Lemma 1. *Suppose* $\gamma = [\partial D]$ *and set* $\partial D = C_0$. *Then there are pairwise disjoint, embedded curves* $C_1, \ldots, C_k$ *in* F *such that*

  (a) $C_0 \cap C_i = \varnothing$ *if* $i \geq 1$,

  (b) *the curves* $C_0, C_1, \ldots, C_k$ *do not setwise separate* F, *and*

  (c) $[C_i]$ *is in the span of* $\{[C_{i+1}], \ldots, [C_k]\}$ *in the closure of one component of* $S^3 - F$ *for* $i = 0, 1, \ldots, k - 1$.[2]

Remark. Regarding (c) above, the component of $S^3 - F$ in which $[C_i]$ is a linear combination of the higher indexed $[C_j]$ can vary with i. A priori, one must check condition (c) on both sides of F for each i.

Proof of Lemma 1. The hypothesis $\theta_F(\gamma, \gamma) = 0$ allows us to assume that D meets F only in transverse circles. The proof now boils down to transforming D into an immersed surface with desirable properties.

Initially, we trade D in for an immersed surface G_0 which satisfies

  (i) $G_0 \cap F$ consists of disjointly embedded, transverse circles,

  (ii) no component of $G_0 \cap F$ separates F,

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[2] As noted by the referee, Lemma 1 remains true if S^3 is replaced by a homology 3-sphere.

(iii) every component of $G_0 \cap F$ separates G_0, and
(iv) $\partial G_0 = \partial D$.

This is accomplished as follows. Suppose C is a curve in $D \cap F$ which separates F. Let F_C denote the closure of one component of $F - C$. We then construct a thickening of F_C in S^3, $F_C \times I$ say, such that $C \times I$ as it lives in $\partial(F_C \times I)$ is an annular neighborhood of C in D. We then replace $C \times I$ in D with the two parallel copies of F_C in $\partial(F_C \times I)$; see Figure 1. Upon performing this operation on all such C, we let G_0 denote the component of the resulting surface which contains ∂D.

The remainder of the proof consists essentially of an innermost circle argument. By property (iii) of G_0, a curve $C \subset G_0 \cap F$ will be called innermost in G_0 if an (open) component of $G_0 - C$ misses F. If C is innermost in G_0, then property (ii) of G_0 implies that C does not separate F. Evidently, if either $C = \partial D$ or the curves $\{\partial G_0, C\}$ setwise separate F, then $\partial G_0 = \partial D$ is null-homologous in the closure of one component of $S^3 - F$—so we are done in either of these cases.

If $C \neq \partial D$ and $\{\partial G_0, C\}$ does not setwise separate F we set $C_k = C$ where k is to be determined. Note that C_k is null-homologous in the closure of one component of $S^3 - F$ and continue the procedure.

Assuming that the procedure must continue, we let $\widetilde{C} \subset [(G_0 \cap F) - C_k]$ denote a curve which is innermost in $G_0 - C_k$. Again, if either $\widetilde{C} = \partial G_0$, $\{\partial G_0, \widetilde{C}\}$ setwise separates F, or $\{\partial G_0, \widetilde{C}, C_k\}$ setwise separates F but $\{\widetilde{C}, C_k\}$ does not setwise separate F, then ∂G_0 is homologous to a multiple (possibly zero) of C_k in the closure of one component of $S^3 - F$. In any of these cases the procedure now stops.

The case where $\{\widetilde{C}, C_k\}$ setwise separates F but $\{\partial G_0, \widetilde{C}\}$ does not setwise separate G_0 is handled in much the same fashion as what occurred in trading D in for G_0. In this case we thicken the closure of a component of $F - \{\widetilde{C}, C_k\}$ and replace an annular neighborhood of $\widetilde{C}$ in G_0 with the complementary portion of the boundary of the thickened component. The resulting surface is called G_1 and contains two curves which are identified with C_k in F, one being C_k and the other arises from the above modification. For simplicity, both curves are labelled C_k in G_1. The procedure, in this case, now continues by considering a curve in $(G_1 \cap F) - \{C_k\}$ which is innermost in $G_1 - \{C_k\}$.

The final possibility of $\widetilde{C}$ is that $\{\partial G_0, \widetilde{C}, C_k\}$ does not setwise separate G_0. In this case, we set $\widetilde{C} = C_{k-1}$ and continue the procedure by considering a curve in $(G_0 \cap F) - \{C_{k-1}, C_k\}$ which is innermost in $G_0 - \{C_{k-1}, C_k\}$. Note further that some component of $G_0 - \{C_{k-1}, C_k\}$ provides a homology which yields $[C_{k-1}] \in \text{span}\{[C_k]\}$ in the closure of one component of $S^3 - F$.
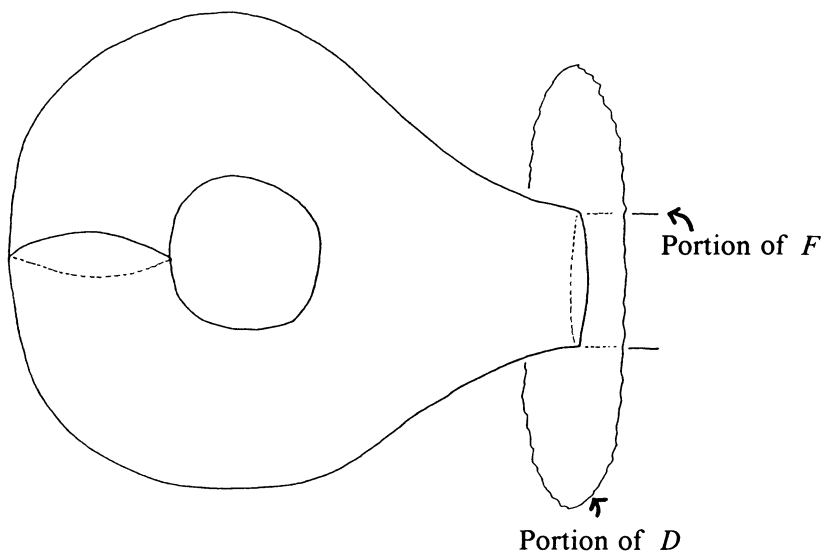
Portion of F

Portion of D

FIGURE 1a
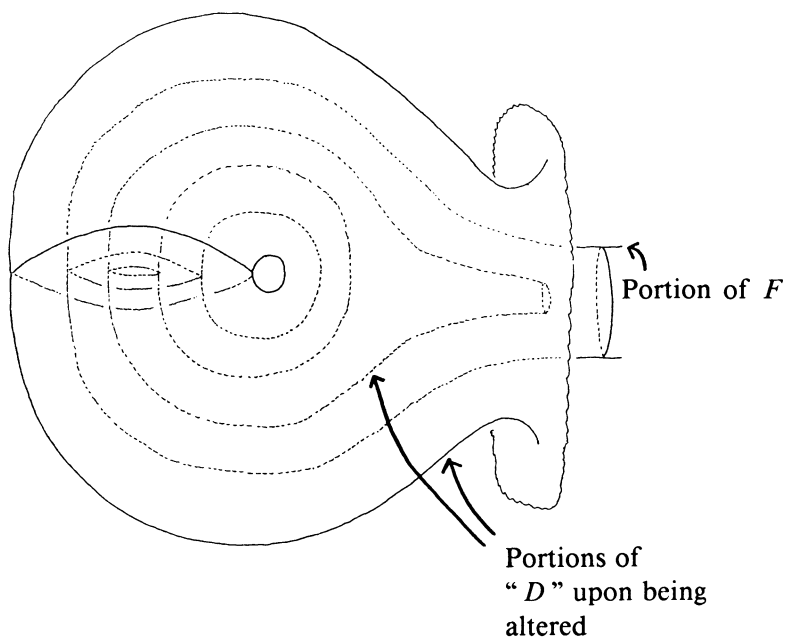

Portion of F

Portions of
" D " upon being
altered

FIGURE 1b

Continuing in this fashion, the process must eventually terminate since $G_0 \cap F$ has only a finite number of components. This completes the proof of Lemma 1. $\square$

Our necessary condition is now obtained by translating the geometric statement of Lemma 1 into an algebraic setting. Upon comparing conditions (a)–(c) of Theorem 1 with those of Lemma 1, it is apparent that this translation merits some discussion. As observed by Meeks and Patrusky, there exist pairs of primitive classes in $H_1(F)$ which are homologically nonintersecting but cannot be realized by a pair of disjointly embedded curves in F [M-P].

Meeks and Patrusky handle this difficulty by employing symplectic matrices. Property (b) of Theorem 1 is a somewhat more naive interpretation of their results—yet is is better suited to our purposes. The translation follows by noting that, since the curves $C_0, \dots, C_k$ in Lemma 1 do not setwise separate F, the class $[C_l] + n_{l+1}[C_{l+1}] + \cdots + n_k[C_k]$ in $H_1(F)$ can be realized by an embedded curve in F and is therefore primitive. This embedded curve is constructed by banding C_l to parallel copies of the $\pm C_j$'s where $j > l$ and $\pm$ refers to orientation. The banding process is discussed in greater detail in §2. Our translation is completed by noting that property (b) of Theorem 1 is stronger than stating that the classes $[C_0], \dots, [C_k]$ are linearly independent in $H_1(F)$. We record the resulting translation as

Proposition 1. *Suppose* $\gamma = [\partial D]$ *and set* $\gamma = \gamma_0$. *Then there exist primitive classes* $\gamma_1, \dots, \gamma_k \in H_1(F)$ *such that*

(a) $\gamma_i \cdot \gamma_j = 0$ *for all* $i, j \in \{0, 1, \dots, k\}$,

(b) $\gamma_i + n_{i+1}\gamma_{i+1} + \cdots + n_k\gamma_k$ *is primitive in* $H_1(F)$ *for each* i *whenever* $n_{i+1}, \dots, n_k \in \mathbf{Z}$, *and*

(c) $\gamma_i \in \text{span}\{\gamma_{i+1}, \dots, \gamma_k\}$, *for each* i, *in the closure of one component of* $S^3 - F$.

Here the henceforth, $\gamma_i \cdot \gamma_j$ denotes the algebraic intersection of γ_i with γ_j in $H_1(F)$.

2. The sufficiency of conditions (a)–(c)

In order to complete the proof of Theorem 1, we need only establish

Proposition 2. *Suppose* $\gamma_0, \gamma_1, \dots, \gamma_k$ *are elements in* $H_1(F)$ *which satisfy properties* (a), (b), *and* (c) *of Theorem* 1. *Then* $\gamma_0 = [\partial D]$.

The last sentence in the statement of Theorem 1 follows from the proof of Proposition 2. Indeed, it provides a terse outline of the proposition's verification, which occupies this entire section.

As one would suspect, we must require that F be standardly embedded in S^3 for Proposition 2 to be valid. In an effort to streamline the proposition's proof, we interpret F being standardly embedded in S^3 to mean that the closures of the components of $S^3 - F$ are genus g handlebodies. Let X_1 and

X_2 denote the closures of the components of $S^3 - F$. By Waldhausen's work on the uniqueness of Heegaard splittings of S^3 [W], we can find systems of properly embedded, disjoint disks $\{\Delta_{r1}\}_{r=1}^{g} \subset X_1$ and $\{\Delta_{s2}\}_{s=1}^{g} \subset X_2$ which satisfy

(i) X_i cut along $\bigcup_r \Delta_{ri}$ is a 3-cell, $i = 1, 2$, and

(ii) $\partial\Delta_{r1}$ meets $\partial\Delta_{s2}$ in precisely δ_{rs} points in a transverse fashion where δ_{rs} denotes the Kronecker delta.

We shall henceforth refer to such collections of disks as symplectically paired cutting disk systems. Note that $\{[\partial\Delta_{r1}]\}$ and $\{[\partial\Delta_{s2}]\}_s$ form a symplectic basis (if properly oriented) for $H_1(F)$ and that the kernel of the induced inclusion $H_1(F) \to H_1(X_j)$ is span$\{[\partial\Delta_{ij}]\}_i$ for $j = 1, 2$.

As suggested by the last sentence of Theorem 1, the proof of Proposition 2 is by induction on k. The usefulness of symplectically paired cutting disk systems is established by

Lemma 2. *Let $\{\Delta_{r1}\}$ and $\{\Delta_{s2}\}$ denote symplectically paired cutting disk systems for X_1, X_2. Then the surface obtained by compressing F along any one of these Δ_{ij} is standardly embedded in S^3.*

The proof of Lemma 2 involves standard handle arguments and will be omitted.

In order to implement Lemma 2 we must show that, given γ_k, we can find symplectically paired cutting disk systems $\{\Delta'_{r1}\}$ and $\{\Delta'_{s2}\}$ such that $\gamma_k = [\partial\Delta'_{ij}]$ for some i, j. That this should be possible follows from γ_k's being null-homologous in either X_1 or X_2.

At this point we recall some basic facts concerning 2-handle slides. Let H denote the orientable, 3-dimensional handlebody of genus g and suppose we attach two 2-handles, h_1 and h_2 say, to H along the curves G_1 and G_2, respectively, in ∂H. If we slide h_1 over h_2, we are in essence replacing the 2-handle h_1 with a new 2-handle h'_1 where the attaching curve of h'_1 is a banded connect sum of G_1 with a parallel copy of $\pm G_2$. The $+$ or $-$ affixed to G_2 is determined by the handle slide and, if $\partial H - \bigcup_i \{G_i\}$ is connected, both can be realized. To see this, the assumption that $\partial H - \bigcup_i \{G_i\}$ is connected allows us to construct another embedded curve $\tau \subset \partial H$ such that $\tau \cap G_1 = \varnothing$ and τ meets G_2 transversely in one point. Moreover, we can find an embedded arc in the closure of $\partial H - \bigcup\{G_1, G_2, \tau\}$ which joins G_1 to τ. We then isotope G_1 along this arc until it just crosses τ. Note now that G_1 can be piped along τ to a parallel translate of G_2 in two distinct directions. One direction will result in a $+$, the other in a $-$.

We are now in a position to establish

Lemma 3. *There exists a symplectically paired cutting disk system $\{\Delta'_{r1}\}$, $\{\Delta'_{s2}\}$ such that $\gamma_k = [\partial\Delta'_{ij}]$ for some i, j.*

Proof. Let $\{\Delta_{r1}\}$, $\{\Delta_{s2}\}$ be an arbitrary symplectically paired cutting disk system and suppose γ_k is null-homologous in X_2. Then $\gamma_k = \sum_s a_s [\partial \Delta_{s2}]$ where $\gcd(a_1, \ldots, a_g) = 1$ since γ_k is primitive.

Let $\{N(\Delta_{s2})\}_s$ denote relative regular neighborhoods of the Δ_{s2}'s in X_2 and view the $N(\Delta_{s2})$'s as 2-handles which are attached to X_1 along the $\partial \Delta_{s2}$'s. Suppose we want to slide $N(\Delta_{i2})$ over $N(\Delta_{j2})$ so that the attaching curve for the resulting 2-handle is either $[\partial \Delta_{i2}] + [\partial \Delta_{j2}]$ or $[\partial \Delta_{i2}] - [\partial \Delta_{j2}]$ on the homology level in $H_1(F)$. We would want to do this in such a way that the Δ_{r1}'s can be altered so that we obtain a new symplectically paired cutting disk system. That this can in fact be accomplished is the heart of the argument.

To this end we initially observe that $F - (\{\partial \Delta_{r1}\} \cup \{\partial \Delta_{s2}\})$ is connected. Hence, by our previous remarks concerning 2-handle slides, we can slide $N(\Delta_{i2})$ over $N(\Delta_{j2})$ in such a fashion so that both $[\partial \Delta_{i2}] \pm [\partial \Delta_{j2}]$ are realizable as the resulting attaching curve. Moreover, the resulting attaching curve misses all the $\partial \Delta_{r1}$'s except $\partial \Delta_{i1}$ and $\partial \Delta_{j1}$ and these exceptions are met transversely in one point each. We may then replace Δ_{j1} by banding it to a parallel copy of Δ_{i1} where the band runs along a subarc in the resulting attaching curve. Thus, we have shown that either of the desired 2-handle slides may be accomplished in such a manner so as to yield a new symplectically paired cutting disk system.

The remainder of the lemma's proof now rests on the following two algebraic facts.

(I) *Suppose* $\gcd(a_1, \ldots, a_g) = 1$. *Then there exists an* $A \in GL(g, \mathbf{Z})$ *such that* $(a_1, \ldots, a_g)$ *is the first row in* A.

(II) *Every element in* $GL(g, \mathbf{Z})$ *can be realized as a finite product of two types of elementary matrices. When acting on the left, the first type corresponds to adding or subtracting one row from another, the second type corresponds to multiplying a row by* -1.

(These are consequences of exercises 7 and 4 respectively on page 180 of [J].)

By (I), γ_k may be realized as the first row of some element in $GL(g, \mathbf{Z})$, A say, with respect to the basis $\{[\partial \Delta_{s2}]\}_s$ for the kernel $H_1(F) \to H_1(X_2)$. By (II), $\{[\partial \Delta_{s2}]\}_s$ can be transformed into A by a sequence of algebraic operations which can be geometrically interpreted as 2-handle slides and reversing the orientations on the core disks of certain 2-handles. Since both of these operations can be performed in such a way so as to recover a new symplectically paired cutting disk system, the lemma follows. $\square$

Combining Lemmas 2 and 3 we obtain the existence of a properly embedded 2-disk D_k in some X_i such that $[\partial D_k] = \gamma_k$ in $H_1(F)$ and if F' is the surface obtained by compressing F along D_k then F' is standardly embedded in S^3. In order to complete the proof of Proposition 2 we must show that $\gamma_0, \ldots, \gamma_{k-1}$ may be viewed as elements in $H_1(F')$ which satisfy properties (a)–(c) with F' replacing F. That $\gamma_0, \ldots, \gamma_{k-1}$ can be so viewed is an immediate consequence of the following lemma.
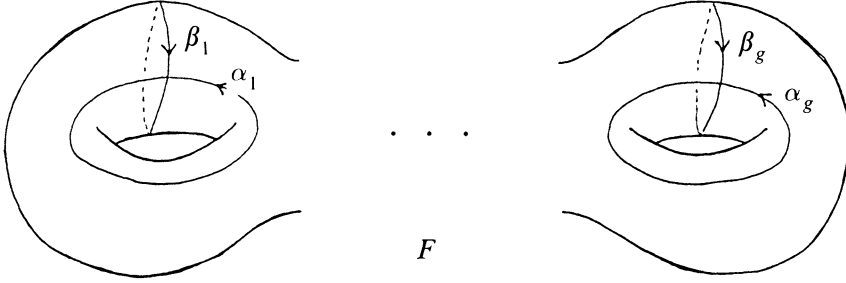
FIGURE 2

Lemma 4. *Let* $\gamma_0, \ldots, \gamma_k \in H_1(F)$ *satisfy properties* (a) *and* (b) *of Proposition 2 and suppose* C_k *is an embedded curve in* F *which represents* γ_k. *Then there exist pairwise disjoint, embedded curves* $C_0, \ldots, C_{k-1} \subset F$ *such that* $C_i \cap C_k = \varnothing$ *for* $i < k$ *and* $\gamma_i = [C_i]$ *in* $H_1(F)$ *for each* $i < k$.

Remarks. Lemma 4 is the converse to the translation from Lemma 1 to Proposition 1. The lemma is due for all practical purposes to Meeks and Patrusky.

Proof of Lemma 4. We proceed via induction on k. Suppose that $\gamma_{l+1}, \ldots, \gamma_k$ are realized by the pairwise disjoint, embedded curves $C_{l+1}, \ldots, C_k$ respectively. We show how to construct C_l.

To this end, let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ denote the symplectic basis for $H_1(F)$ depicted in Figure 2. (We will abuse notation and identify these classes with their representative curves.) Now let $f: F \to F$ be a diffeomorphism which carries C_j onto α_j for $j = l + 1, \ldots, k$.

Write $f_*(\gamma_l) = \sum_{i=1}^{g}(a_i\alpha_i + b_i\beta_i)$. Since $f_*(\gamma_i) \cdot f_*(\gamma_j) = \gamma_i \cdot \gamma_j$, property (a) implies that $b_{l+1}, \ldots, b_k = 0$. Property (b) implies that

$$f_*\left(\gamma_l - \sum_{i=l+1}^{k} a_i\gamma_i\right) = f_*(\gamma_l) - \sum_{i=l+1}^{k} a_i\alpha_i$$

$$= \sum_{i=1}^{l}(a_i\alpha_i + b_i\beta_i)$$

is primitive in $H_1(F)$. It follows that $f_*(\gamma_l - \sum_{i=l+1}^{k} a_i\gamma_i)$ can be realized in the surface obtained by compressing F along $\alpha_{l+1}, \ldots, \alpha_k$ by a simple closed curve. By general position, this curve may be assumed to live in $F - \{\alpha_{l+1}, \ldots, \alpha_k\}$. Denote this curve by C'.

Clearly, $[C']$, $\alpha_{l+1}, \ldots, \alpha_k$ are linearly independent in $H_1(F)$. This implies that the curves C', $\alpha_{l+1}, \ldots, \alpha_k$ do not setwise separate F. Using the techniques of Lemma 3, we can band C' to parallel copies of the $\alpha_i, l + 1 \le i \le k$, so as to obtain an embedded curve C which represents $f_*(\gamma_l)$ and misses $\alpha_{l+1}, \ldots, \alpha_k$. We take $C_l = f^{-1}(C)$. $\square$

We can therefore view $\gamma_0, \ldots, \gamma_{k-1}$ in $H_1(F')$ as the classes represented by the curves $C_0, \ldots, C_{k-1}$ where C_k is taken to be ∂D_k. In order to verify that these classes satisfy property (c) with respect to F' one simply realizes the homologies guaranteed by property (c) with respect to F by immersed oriented surfaces in X_1 and X_2. These surfaces are assumed to meet F in the curves $C_0, \ldots, C_k$. The required homologies in the closures of the components of $S^3 - F'$ are realized by immersed surfaces which are obtained, essentially, from the above immersed surfaces by cut and paste modifications. This completes the verification of the induction step for Proposition 2.

Inductively, there exist disjointly embedded disks $D_0, \ldots, D_{k-1}$ with $\partial D_i \subset F'$ and $[\partial D_i] = \gamma_i \in H_1(F')$. There remains a minor problem, this being that $[\partial D_i] \neq \gamma_i$ in $H_1(F)$. We do know however that $[C_i] = \gamma_i$ in $H_1(F)$. Since $C_0, \ldots, C_k$ do not separate $H_1(F)$, we can construct an arc A in F' which misses $C_0, \ldots, C_{k-1}$ and joins the pair of copies of C_k we see in F' after compressing F along D_k. One now simply isotopes the ∂D_i, $i = D, \ldots, k-1$ in F' so as to miss A together with the copies of D_k in F'. Then $[\partial D_i] = \gamma_i$ in $H_1(F)$.

This completes the proof of Proposition 2 and hence completes the proof of Theorem 1.

Remark. It should be noted that although the proof of Theorem 1 constructs embedded disks whose boundaries realize every admissible homology class in F, there are homotopy classes in F which are realized by unknotted circles (whose self-paring under the Seifert form is trivial) but cannot be obtained via Theorem 1's proof. Consider, for example, the tunnel number one knots.

3. SOME APPLICATIONS

We move immediately into the

Proof of Theorem 2. We are initially given a symplectic basis $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ for $H_1(F)$ such that the α_i's form a basis for $\ker(H_1(F) \to H_1(X_1))$ and the β_i's form a basis for $\ker(H_1(F) \to H_1(X_2))$. The symplectic restriction implies that the intersection pairing on $H_1(F)$ with respect to this basis is a direct sum of g copies of $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The additional kernel restrictions on this basis imply that the Seifert form on F with respect to this basis is a direct sum of g copies of $\begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$.

Now suppose $\gamma = \sum_i (a_i \alpha_i + b_i \beta_i)$ where $\theta_F(\gamma, \gamma) = 0$ and $\gcd(a_1, \ldots, a_g) = 1$ say. If each $b_i = 0$, it follows that $\gamma = [\partial D]$ by Theorem 1. So suppose that some $b_i \neq 0$. We can then find a primitive class γ_1 of the form $\gamma_1 = \sum_i b_i' \beta_i$ such that $n(\sum_i b_i' \beta_i) = \sum_i b_i \beta_i$, for some nonzero integer n. Clearly $\gamma \in \mathrm{span}\{\gamma_1\}$ in $H_1(X_1)$ and $\gamma + m\gamma_1$ is primitive for all $m \in \mathbf{Z}$.

We are left with showing that $\gamma \cdot \gamma_1 = 0$. Since $\theta_F(\gamma, \gamma) = 0$, it follows that $-\sum_i a_i b_i = 0$ and hence $\sum_i a_i b_i' = 0$. But this implies that $\gamma \cdot \gamma_1 = 0$. Therefore γ and γ_1 satisfy properties (a)–(c) of Theorem 1 and so $\gamma =$

$[\partial D]$. The case $\gcd(b_1, \ldots, b_g) = 1$ follows similarly. Thus, the additional gcd condition on γ is sufficient for $\gamma = [\partial D]$.

As for the converse direction, suppose $\gamma = [\partial D]$. By Theorem 1 we can then find classes $\gamma = \gamma_0, \gamma_1, \ldots, \gamma_k$ in $H_1(F)$ which satisfy properties (a)–(c). Write $\gamma_i = \gamma_{i\alpha} + \gamma_{i\beta}$ where $\gamma_{i\alpha} \in \mathrm{span}\{\alpha_j\}_j$ and $\gamma_{i\beta} \in \mathrm{span}\{\beta_j\}_j$ for $i = 0, \ldots, k$.

Clearly $\gamma_k = \gamma_{k\alpha}$ or $\gamma_k = \gamma_{k\beta}$. Hence $\mathrm{span}\{\gamma_k\} = \mathrm{span}\{\gamma_{kx_k}\}$ where $x_k \in \{\alpha, \beta\}$ and γ_{kx_k} is primitive by (b). Inductively we assume that

$$\mathrm{span}\{\gamma_{l+1}, \ldots, \gamma_k\} = \mathrm{span}\{\gamma_{l+1x_{l+1}}, \ldots, \gamma_{kx_k}\}$$

in $H_1(F)$ where $x_i \in \{\alpha, \beta\}$ and γ_{ix_i} is primitive for $i = l+1, \ldots, k$. Now, if either $\gamma_{l\alpha}$ or $\gamma_{l\beta}$ is trivial, the induction step is evidently completed. If, both $\gamma_{l\alpha}$ and $\gamma_{l\beta}$ are nontrivial, we apply (c) to obtain either $\gamma_{l\alpha}$ or $\gamma_{l\beta}$ is in

$$\mathrm{span}\{\gamma_{l+1}, \ldots, \gamma_k\} = \mathrm{span}\{\gamma_{l+1x_{l+1}}, \ldots, \gamma_{kx_k}\}$$

in the closure of one component of $S^3 - F$. This is of course due to the fact that either all the α_i's or β_i's are trivial depending on the component of $S^3 - F$.

Let us suppose that this homology occurs in X_2—so all the β_i's vanish. Then $\gamma_{l\alpha}$ is a linear combination of the γ_{ix_i}'s where $x_i = \alpha$ in X_2. We note however that this linear combination being $\gamma_{l\alpha}$ remains valid in F. The geometric reason for this is that all the $\gamma_{i\alpha}$'s bound immersed oriented surfaces in X_1, so if the immersed oriented surface in X_2 which realizes this linear combination were to algebraically intersect a spine for X_2, realized by a bouquet of circles, in a nontrivial fashion we would have $H_1(S^3) \neq 0$. Hence, the immersed surface in X_2 can be made to geometrically miss the spine of X_2—and therefore can be pushed into F. It follows that

$$\gamma_{l\alpha} \in \mathrm{span}\{\gamma_{l+1x_{l+1}}, \ldots, \gamma_{kx_k}\} = \mathrm{span}\{\gamma_{l+1}, \ldots, \gamma_k\}$$

in $H_1(F)$. It follows by (b) then that $\gamma_{l\beta}$ must be primitive and

$$\mathrm{span}\{\gamma_{l\beta}, \ldots, \gamma_{kx_k}\} = \mathrm{span}\{\gamma_l, \ldots, \gamma_k\}$$

in $H_1(F)$.

This argument is clearly symmetric in α and β. Therefore the converse holds. $\square$

We proceed into the

Proof of Corollary 1. Using the above notation we write $\delta = \delta_\alpha + \delta_\beta$ and assume $\delta = [\partial D]$. Let $P \in \partial D$ and Π denote a plane in $\mathbf{R}^3$ which misses $F \cup D$. We assume, moreover, that there is a line segment joining P to Π whose interior misses F. By an isotopy (in S^3) we can assume further that the interior of this line segment also misses D. We now reflect $F \cup D$ across Π and join the F's together with a tube running along this line segment and its reflection across Π. We may then band D and its reflection together along this tube—obtaining D'. The corollary now follows by noting that $\gamma = [\partial D']$ satisfies $\theta_{F \cup r(F)}(\gamma, \gamma) = 0$ and γ_α or γ_β is primitive if and only if δ_α or δ_β is primitive, respectively. $\square$

This is an appropriate time to present some examples. For this purpose we will refer to the symplectic basis depicted in Figure 2. If γ is primitive and $\theta_F(\gamma, \gamma) = 0$, then we may decompose γ as $\gamma = \gamma_\alpha + \gamma_\beta$ as in Theorem 2. By further decomposing γ we have $\gamma = m\gamma'_\alpha + n\gamma'_\beta$ where γ'_α and γ'_β are either primitive or trivial and $\gcd(m, n) = 1$. As in the proof of Theorem 2 $\gamma'_\alpha \cdot \gamma'_\beta = 0$. Thus, every primitive γ which satisfies $\theta_F(\gamma, \gamma) = 0$ and is not null-homogeneous in the closure of one component of $S^3 - F$ is of the form $\gamma = m\gamma'_\alpha + n\gamma'_\beta$ where γ'_α and γ'_β are primitive, $\gamma'_\alpha \cdot \gamma'_\beta = 0$ and $\gcd(m, n) = 1$. For γ to be realized by an unknotted circle in S^3 we must have $|m| = 1$ or $|n| = 1$. Thus, if γ is primitive and $\theta_F(\gamma, \gamma) = 0$, the additional hypothesis that $\gamma = [\partial D]$ may be viewed as further restricting γ by one degree of freedom.

As for a specific example, if $\text{genus}(F) \geq 2$ then take $\gamma'_\alpha = \alpha_1$ and $\gamma'_\beta = \beta_2$. Clearly $\gamma'_\alpha \cdot \gamma'_\beta = 0$ but $\gamma = 2\gamma'_\alpha + 3\gamma'_\beta$ can never be realized by an unknotted circle in S^3. This class can, however, be realized by a ribbon knot by banding two copies of α_1 together with three copies of β_2 in the obvious way. In fact, this is the case in general.

Corollary 2. *Suppose* $\gamma \in H_1(F)$ *such that* γ *is primitive and* $\theta_F(\gamma, \gamma) = 0$. *Then* γ *can be realized by a ribbon knot.*

Proof. Write $\gamma = m\gamma'_\alpha + n\gamma'_\beta$ as above. If either $|m| \leq 1$ or $|n| \leq 1$ then γ can be realized by an unknot by Theorem 2. If both $|m| > 1$ and $|n| > 1$, then $\theta_F(\gamma, \gamma) = 0$ implies that $\gamma'_\alpha \cdot \gamma'_\beta = 0$ and hence we apply Theorem 1 to find disjointly embedded disks D_α and D_β such that $\gamma'_\alpha = [\partial D_\alpha]$ and $\gamma'_\beta = [\partial D_\beta]$. Using essentially Lemma 3 of §2, γ can be realized via a sequence of banding operations involving ∂D_α and ∂D_β and banded connect sums of these curves. The union of these bands together with parallel copies of D_α and D_β form an immersed disk in S^3 which, if constructed with reasonable care, has only ribbon singularities. Hence the boundary of this immersed disk is a ribbon knot which realizes γ.

Remarks. (1) That the disk described above is immersed and not embedded is geometrically due to D_α and D_β not being in a single component of $S^3 - F$ initially. (Of course, Theorem 2 states that it cannot be embedded.)

(2) It should be noted that not every class in $H_1(F)$ can be realized by a ribbon knot. For instance, $3\alpha_1 + 2\beta_1$ on the standardly embedded torus can only be realized by a trefoil.

Theorem 2 may be restated as

Corollary 3. *Let* F *be standardly embedded in* $\mathbf{R}^3$ *and suppose* $f_t : \mathbf{R}^3 \to \mathbf{R}^3$ *is a smooth isotopy with* $f_0 = \text{id}$ *such that* $f_1(\mathbf{R}^2)$ *meets* F *in transverse circles. Then each of these circles represents either the trivial element in* $H_1(F)$ *or satisfies the strongly primitive criterion of Theorem 2. Conversely, if* $\gamma \in H_1(F)$ *satisfies* $\theta_F(\gamma, \gamma) = 0$ *and the strongly primitive criterion of Theorem 2, then*

there is an isotopy $f_t: \mathbf{R}^3 \to \mathbf{R}^3$, *as above, such that* γ *is realized by a component of* $f_1(\mathbf{R}^2) \cap F$.

Reversing these isotopies, it follows that Theorem 2 restricts how F can be isotoped in $\mathbf{R}^3$.

We now enter into the

Proof of Theorem 3. Let $\alpha_1, \beta_1, \ldots, \alpha_g, \beta_g$ denote the symplectic basis of $H_1(F)$ depicted in Figure 2 and set

$$\langle \alpha \rangle = \mathrm{span}\{\alpha_1, \ldots, \alpha_g\}, \qquad \langle \beta \rangle = \mathrm{span}\{\beta_1, \ldots, \beta_g\}.$$

Denoting $[C'_i]$ by γ'_i for $i = 1, \ldots, g$, we have by analogy

$$\langle \gamma' \rangle = \mathrm{span}\{\gamma'_1, \ldots, \gamma'_g\}.$$

We claim that either $\langle \alpha \rangle \cap \langle \gamma' \rangle \neq 0$ or $\langle \beta \rangle \cap \langle \gamma' \rangle \neq 0$. To this end, suppose $\langle \alpha \rangle \cap \langle \gamma' \rangle = 0$ and let $\pi: H_1(F) \to \langle \alpha \rangle$ denote the obvious projection. (Note that $H_1(F) = \langle \alpha \rangle \oplus \langle \beta \rangle$.) Since $\ker \pi = \langle \beta \rangle$, the claim follows upon verifying that $\ker(\pi|\langle \gamma' \rangle)$ is nontrivial.

Suppose to the contrary that $\ker(\pi|\langle \gamma' \rangle) = 0$. Then $\{\pi(\gamma'_1), \ldots, \pi(\gamma'_g)\}$ is a linearly independent subset in $\langle \alpha \rangle$. It follows that $\langle \alpha \rangle / \pi(\langle \gamma' \rangle)$ is a torsion module—for otherwise $\bigoplus_g \mathbf{Z}$ would contain a submodule of rank exceeding g. It follows that there exist positive integers n_i such that $n_i \alpha_i \in \pi(\langle \gamma' \rangle)$ for $i = 1, \ldots, g$. Since $\ker \pi = \langle \beta \rangle$, there exist $\gamma^*_i \in \langle \gamma' \rangle$ such that $\gamma^*_i = n_i \alpha_i + \sum_{j=1}^g b_{ij} \beta_j$ for $i = 1, \ldots, g$.

By hypothesis, $\theta_F(\gamma'_r, \gamma'_s) = 0$ for all r and s. The bilinearity of θ_F implies that $\theta_F(\gamma^*_r, \gamma^*_s) = 0$ for all r and s. But $\theta_F(\gamma^*_r, \gamma^*_s) = -n_r b_{sr}$ for each pair of r and s. Since $n_r > 0$ for each r, it follows that $b_{sr} = 0$ for all r and s. This implies that $\langle \gamma' \rangle \cap \langle \alpha \rangle \neq 0$—a contradiction. Therefore $\ker(\pi|\langle \gamma' \rangle)$ must be nontrivial and the claim follows.

Suppose that $\langle \alpha \rangle \cap \langle \gamma' \rangle \neq 0$. Then there exist integers, not all equal to zero, m_i, such that $\sum_{i=1}^g m_i \gamma'_i \in \langle \alpha \rangle$. Since $H_1(F)$ is free, we can assume that $\gcd(m_1, \ldots, m_g) = 1$. Applying the algebraic facts in the proof of Lemma 3 to the curves $C'_1, \ldots, C'_g$ (viewed as attaching spheres of 2-handles), we can trade the C'_i in for $\{C''_{11}, \ldots, C''_{1g}\}$ where the C''_{ij} do not setwise separate F, $\mathrm{span}\{[C'_1], \ldots, [C'_g]\} = \mathrm{span}\{[C''_{11}], \ldots, [C''_{1g}]\}$, and $[C''_{1g}] = \sum_{i=1}^g m_i \gamma' \in \langle \alpha \rangle$.

On the other hand, $[C''_{1g}]$ is certainly primitive and null-homologous in the closure of one component of $S^3 - E$. Lemmas 2 and 3 may therefore be applied to a symplectically paired cutting disk system to yield a properly embedded 2-disk, D_g say, in the closure of one component of $S^3 - F$ such that $[\partial D_g] = [C''_{1g}]$ and F cut along D_g is a standardly embedded surface of genus $g - 1$.

Let $\partial D_g = C_{1g}$. Applying Lemma 4, there exist disjointly embedded curves $C_{11}, \ldots, C_{1g-1}$ in F such that $C_{1j} \cap C_{1g} = \varnothing$, $j < g$, $C_{11}, \ldots, C_{1g}$ do not

setwise separate F and $[C_{1j}] = [C_{1j}'']$ for all j. We now cut F along D_g and apply induction. In doing so, it should be noted that the C_{1j}'s, $1 \le j \le g - 1$, now play the role of the original C_i''s.

The induction step tells us that the C_{1j}'s can be banded together in some fashion corresponding to 2-handle slides to obtain curves $C_{21}, \ldots, C_{2g-1}$ in F_{g-1} and embedded disks $D_1, \ldots, D_{g-1}$ in S^3 such that $\partial D_i \subset F_{g-1}$ for all i and $[C_{2j}] = [\partial D_j]$ in $H_1(F_{g-1})$ for all j. We take $C_i = \partial D_i$ for $i = 1, \ldots, g$ and note that the banding operations performed to the C_{1j}'s in obtaining the C_{2j}'s can be mimicked by the C_{ij}''s and hence by the C_i's in order to obtain the curves C_i'' as in the statement of Theorem 3. (Note that $C_g'' = C_{1g}''$.) $\square$

4. Final remarks

As noted in the introduction, there is a striking difference between closed surfaces and Seifert surfaces with regards to the problem of realizing homology classes by unknotted circles. This was unexpected. In fact, the original hope was that knowledge of the homology classes on a closed surface which were realizable by unknots would have a more immediate impact on the problem of determining when a knot is ribbon when taken in conjunction with Corollary 3 of [T].

There is, however, something of a round about way in which Theorem 2 can be applied to Seifert surfaces—at least to show a homology class on a Seifert surface is not realizable by an unknot. The idea is to embed the Seifert surface in some F so that the homology class in question is not strongly primitive in $H_1(F)$. The viability of this approach is of course dependent on one's ability to construct such an embedding. However, interesting examples can be conjured up by working in F and employing Corollary 2.

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